ALGEBRAS, GROUPS AND GEOMETRIES 40 135 – 152 (2024)
DOI 10.29083/AGG.40.02.2024/SC135

RINGS IN WHICH ELEMENTS ARE A SUM OF A CENTRAL AND NILPOTENT ELEMENT

Yosum Kurtulmaz
Department of Mathematics, Bilkent University
Ankara, Turkey
kurtulmazy@gmail.com

Abdullah Harmanci
Department of Mathematics, Hacettepe University
Ankara, Turkey
harmanci@hacettepe.edu.tr

Received May 25, 2024

Abstract

In this paper, we introduce a new class of rings whose elements are a sum of a central element and a nilpotent element, namely, a ring R is called CN if each element a of R has a decomposition $a = c + n$ where c is central and n is nilpotent. In this note, we characterize elements in $M_n(R)$ and $U_2(R)$ having CN-decompositions. For any field F, we give examples to show that $M_n(F)$ can not be a CN-ring. For a division ring D, we prove that if $M_n(D)$ is a CN-ring, then the cardinality of the center of D is strictly greater than n. Especially, we investigate several kinds of conditions under which some subrings of full matrix rings over CN rings are CN.

2010 AMS Mathematics Subject Classification: 15B33, 15B36, 16S70, 16U99.

Key words and phrases: CN ring, CN decomposition, matrix ring, uniquely nil clean ring, uniquely nil*-clean ring.

1 Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $\mathrm{Inv}(R)$, $J(R)$, $C(R)$ and $\mathrm{nil}(R)$ will denote the group of units, the Jacobson radical, the center and the set of all nilpotent elements of a ring R, respectively. Recall that in [2], uniquely nil clean rings are defined. An element a in a ring R is called *uniquely nil clean* if there is a unique idempotent $e \in R$ such that $a - e$ is nilpotent. The ring R is *uniquely nil clean* if each of its elements is uniquely clean. It is proved that in a uniquely nil clean ring, every idempotent is central. Also a uniquely nil clean ring R is called *uniquely strongly nil clean* [5] if a and e commute. Strongly nil cleanness and uniquely strongly nil cleanness are equivalent by [2]. Let R be a (*)-ring. In [7], $a \in R$ is called *uniquely strongly nil *-clean ring* if there is a unique projection $p \in R$, i.e., $p^2 = p = p^*$, and $n \in \mathrm{nil}(R)$ such that $a = p + n$ and $pn = np$. R is called a *uniquely strongly nil *-clean ring* if each of its elements is uniquely strongly nil *-clean. Another version of the notion of clean rings is that of CU rings. In [1], an element $a \in R$ is called a *CU element* if there exist $c \in C(R)$ and $n \in \mathrm{nil}(R)$ such that $a = c + n$. The ring R is called *CU* if each of its elements is CU. Motivated by these facts, we investigate basic properties of rings in which every element is the sum of a central element and a nilpotent element.

In what follows, $\mathbb{Z}_n$ is the ring of integers modulo n for some positive integer n. Let $M_n(R)$ denote the full matrix ring over R and $U_n(R)$ stand for the subring of $M_n(R)$ consisting of all $n \times n$ upper triangular matrices. And in the following, we give definitions of some other subrings of $U_n(R)$ to discuss in the sequel whether they satisfy CN property:

$$D_n(R) = \{(a_{ij}) \in M_n(R) \mid \text{ all diagonal entries of } (a_{ij}) \text{ are equal}\},$$

$$V_n(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{n} a_j e_{(i-j+1)i} \mid a_j \in R \right\},$$

$$V_n^k(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{k} x_j e_{(i-j+1)i} + \sum_{i=j}^{n-k} \sum_{j=1}^{n-k} a_{ij} e_{j(k+i)} : x_j, a_{ij} \in R \right\}$$

where $x_i \in R$, $a_{js} \in R$, $1 \leq i \leq k$, $1 \leq j \leq n-k$ and $k+1 \leq s \leq n$,

$$D_n^k(R) = \{\left\{\sum_{i=1}^{k}\sum_{j=k+1}^{n} a_{ij}e_{ij} + \sum_{j=k+2}^{n} b_{(k+1)j}e_{(k+1)j} + cI_n \mid a_{ij}, b_{ij}, c \in R\right\}$$

where $k = [n/2]$, i.e., k satisfies $n = 2k$ when n is an even integer, and $n = 2k+1$ when n is an odd integer, and

$$D_n^{\mathbb{Z}}(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = a_{nn} \in \mathbb{Z}, a_{ij} \in R, \{i,j\} \subseteq \{2,3,\ldots,n-1\}\}.$$

2 Basic Properties and Examples

Definition 2.1. Let R be a ring with identity. An element $a \in R$ is called *CN* or *it has a CN-decomposition* if $a = c + n$, where $c \in C(R)$ and $n \in$ nil(R). If every element of R has a CN decomposition, then R is called *a CN ring.*

We present some examples to illustrate the concept of CN property for rings.

Example 2.2. (1) Every commutative ring is CN.

(2) Every nilpotent element in a ring R has a CN decomposition.

(3) For a field F and for any positive integer n, $D_n(F)$ is a CN ring.

Proposition 2.3. *Let R be a ring and n a positive integer. Then $A \in M_n(R)$ has a CN decomposition if and only if for each $P \in GL_n(R)$, $PAP^{-1} \in M_n(R)$ has a CN decomposition.*

Proof. Assume that $A \in M_n(R)$ has a CN decomposition $A = C + N$ where $C \in C(M_n(R))$ and $N \in$nil$(M_n(R))$. Then $PAP^{-1} = PCP^{-1} + PNP^{-1}$ is a CN decomposition of PAP^{-1} since $PCP^{-1} = C \in C(M_n(R))$ and it is obvious that $PNP^{-1} \in$ nil$(M_n(R))$. Conversely, suppose that PAP^{-1} has a CN decomposition $PAP^{-1} = C + N$. Then $A = P^{-1}CP + P^{-1}NP$ is the CN decomposition of PAP^{-1}. $\square$

Let R be a commutative ring and n a positive integer. The following result gives us a way to find out whether $A \in M_n(R)$ has a CN decomposition. Note that it is easily shown that for a commutative ring $A \in C(M_n(R))$ if and only if $A = cI_n$ for some $c \in R$.

Theorem 2.4. *Let R be a commutative ring. Then $A \in M_n(R)$ has a CN decomposition if and only if $A - cI_n \in nil(M_n(R))$ for some $c \in R$.*

Proof. Assume that $A \in M_n(R)$ has a CN decomposition. By assumption there exists $c \in R$ such that $A - cI_n \in \text{nil}(M_n(R))$. Conversely, suppose that for any $A \in M_n(R)$, there exists $c \in R$ such that $A - cI_n \in \text{nil}(M_n(R))$. Since cI_n is central in $M_n(R)$, $A \in M_n(R)$ has a CN decomposition. □

Remark. Let R be a commutative ring. Then $A \in M_n(R)$ is a nilpotent matrix if and only if all eigenvalues of A are zero. A ring R is *reduced* if R has no nonzero nilpotent element. Hence we have.

Corollary 2.5. *Let R be a commutative reduced ring and n a positive integer. Then $A \in M_n(R)$ has a CN decomposition if and only if the only eigenvalue for $A - cI_n$ is 0 for some $c \in R$.*

Proposition 2.6. *Let R be a commutative ring. Then $U_2(R)$ is a CN ring if and only if for any $a, b \in R$, there exists $c \in R$ such that $a - c$, $b - c \in nil(R)$.*

Proof. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(R)$ has CN decomposition if and only if there exist $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C(M_2(R))$ and $N = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in \text{nil}(M_2(\mathbb{R}))$ such that $A = C + N$. Since $N \in \text{nil}(M_2(R))$ if and only if x, $z \in \text{nil}(R)$, $A = C + N$ is the CN decomposition of A if and only if there exists $c \in R$ such that $A - cI \in \text{nil}(M_2(R))$ if and only if $a - c$, $b - c \in \text{nil}(R)$. □

Example 2.7. Let $R = \mathbb{Z}$ and $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \in U_2(R)$. Then there is no $c \in \mathbb{Z}$ such that $3 - c$ and $5 - c$ are nilpotent. By Proposition 2.6, $U_2(\mathbb{Z})$ is not CN.

Theorem 2.8. *Let R be a commutative local ring. If $M_2(R)$ is a CN ring, then $R/J(R)$ is not isomorphic to $\mathbb{Z}_2$.*

Proof. Assume that $M_2(R)$ is a CN ring. Suppose that $R/J(R)$ is isomorphic to $\mathbb{Z}_2$ and we get a contradiction. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$ and $f(c) = det(A - cI_2)$ be the characteristic polynomial of A. Then $f(c) = c(c-1) \in \text{nil}(R)$. By Proposition 2.6, $1-c$ and c are nilpotent. Since $1 = c + (1-c)$, By hypothesis, c or $1-c$ is invertible, therefore $c \in J(R)$ or $1 - c \in J(R)$. This is a contradiction. $\square$

In [1], Chen and at al. defined and studied *CU* rings. Let R be a ring. An element $a \in R$ has a *CU*-decomposition if $a = c + u$ for some $c \in C(R)$ and $u \in U(R)$. A ring R is called *CU*, if every element of R has a *CU*-decomposition.

Proposition 2.9. *Every CN ring is CU.*

Proof. Let R be a CN ring and $a \in R$. By assumption $a + 1 = c + n$ for some $c \in C(R)$ and $n \in N(R)$. Hence $a = c+(n-1)$ is a *CU* decomposition of a. $\square$

Theorem 2.10. *Let R be a division ring and n a positive integer. If $M_n(R)$ is a CN ring, then $|C(R)| > n$.*

Proof. Assume that $|C(R)| < n$. Consider A as a diagonal matrix which has the property that each element of $C(R)$ is one of the diagonal entries of A. For such a matrix A there is no $c \in C(R)$ for which $A - cI$ is a unit. Hence $M_n(R)$ is not a CU ring. By Proposition 2.9, $M_n(R)$ can not be a CN ring. This contradicts hypothesis. So $|C(R)| > n$. $\square$

The converse of Proposition 2.9 is not true in general.

Example 2.11. Let $\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$ be the ring of real quaternions, where $i^2 = j^2 = k^2 = ijk = 1$ and $ij = -ji, ik = -ki, jk = -kj$. $\mathbb{H}$ is a noncommutative division ring. Note that $C(\mathbb{H}) = \mathbb{R}$ and $nil(\mathbb{H}) = 0$. Let $a \in \mathbb{H}$. If $a = 0$, then $0 = 1 + (-1)$ is the *CU*-decomposition. If $a \neq 0$, then $a = 0 + a$ is the *CU*-decomposition of a. Hence $\mathbb{H}$ is a *CU* ring. On the other hand there is no CN decomposition of $i \in \mathbb{H}$. Hence it is not a CN ring.

Example 2.12. Let D be a division ring and consider the ring $D_2(D)$. The ring $D_2(D)$ is a noncommutative local ring, and so it is a CU-ring, but not a CN ring.

For a positive integer n, one may suspect that if R is a *CN* ring then the matrix ring $M_n(R)$ is also *CN*. The following examples shows that this is not true in general. Also whether or not $M_n(R)$ to be a CN ring does not depend on the cardinality of $C(R)$ comparing with n, that is, $|C(R)| \geq n$ or $|C(R)| < n$.

Example 2.13. (1) Since $\mathbb{Z}$ is commutative, it is a CN ring. But $R = M_2(\mathbb{Z})$ is not a CN ring.
(2) $R = M_2(\mathbb{Z}_3)$ is not a CN ring.
(3) $R = M_3(\mathbb{Z}_2)$ is not a CN ring.

Proof. (1) Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in M_2(\mathbb{Z})$ which is neither central nor nilpotent. Let $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \in C(M_2(\mathbb{Z}))$ and $N = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in \mathrm{nil}(M_2(\mathbb{Z}))$ with $A = C + N$. Then $x + t = 0$ and $zy = xt$. This is a contradiction. Hence A does not have CN decomposition.
(2) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$ which is neither central nor nilpotent. Assume that A has CN decomposition with $A = C + N$ where $C = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in C(M_2(\mathbb{Z}_3))$ and $N = \begin{bmatrix} x & y \\ t & u \end{bmatrix} \in \mathrm{nil}(M_2(\mathbb{Z}_3))$. $A = C + N$ implies $1 = a + x$, $0 = a + u$ and $y = t = 0$. These equalities do not satisfied in $\mathbb{Z}_3$. For if $a = 0$, then $x = 1$; if $a = 1$, then $x = 0$ and $u = 2$; if $a = 2$, then $x = 2$ and $u = 1$. All these lead us a contradition. Hence $M_2(\mathbb{Z}_3)$ is not a CN ring.
(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_3(\mathbb{Z}_2)$ which is neither central nor nilpotent. Assume that A has CN decomposition with $A = C + N$ where $C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in C(M_3(\mathbb{Z}_2))$ and $N = \begin{bmatrix} x & y & z \\ t & u & v \\ k & l & m \end{bmatrix} \in \mathrm{nil}(M_3(\mathbb{Z}_2))$. $A = C + N$

implies $1 = a + x$, $0 = a + u$, $0 = a + m$ and $y = z = v = t = k = l = 0$. These equalities do not satisfied in $\mathbb{Z}_2$. Hence $M_3(\mathbb{Z}_2)$ is not a CN ring. In fact, assume that $1 = a + x$ holds in $\mathbb{Z}_2$. There are two cases for a. $a = 0$ or $a = 1$. If $a = 1$ then $x = 0$ and $u = 1$. N being nilpotent implies $u = 1$ is nilpotent. A contradiction. Otherwise, $a = 0$. Then $x = 1$. Again N being nilpotent implies $x = 1$ is nilpotent. A contradiction. Thus $M_3(\mathbb{Z}_2)$ is not a CN ring. $\square$

In spite of the fact that $U_n(R)$ need not be *CN* for any positive integer n, there are *CN* subrings of $U_n(R)$.

Proposition 2.14. *For a ring R and an integer $n \geq 1$, the following are equivalent:*
(1) R is CN.
(2) $D_n(R)$ is CN.
(3) $D_n^k(R)$ is CN.
(4) $V_n(R)$ is CN.
(5) $V_n^k(R)$ is CN.

Proof. Note that the elements of $D_n(R)$, $D_n^k(R)$, $V_n(R)$ and $V_n^k(R)$ having zero as diagonal entries are nilpotent. To complete the proof, it is enough to show (1) holds if and only if (2) holds for $n = 4$. The other cases are just a repetition.

(1) $\Rightarrow$ (2) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_5 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & a_1 \end{bmatrix} \in D_4(R)$. By (1), there exist $c \in C(R)$ and $n \in \text{nil}(R)$ such that $a_1 = c + n$.

Let $C = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$ and $N = \begin{bmatrix} n & a_2 & a_3 & a_4 \\ 0 & n & a_5 & a_6 \\ 0 & 0 & n & a_7 \\ 0 & 0 & 0 & n \end{bmatrix}$. Then $C \in C(V_n(R))$ and $N \in \text{nil}(D_n(R))$.

(2) $\Rightarrow$ (1) Let $a \in R$. By (2) $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \in D_4(R)$ has a CN

decomposition $A = C + N$ where $C = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \in C(D_4(R))$ and

$N = \begin{bmatrix} n & * & * & * \\ 0 & n & * & * \\ 0 & 0 & n & * \\ 0 & 0 & 0 & n \end{bmatrix} \in C(D_n(R))$. Then $a = c + n$ with $c \in C(R)$ and $n \in$ nil(R). □

Lemma 2.15. *Every homomorphic image of CN ring is CN ring.*

Proof. Let $f : R \to S$ be an epimorphism of rings with R CN ring. Let $s = f(x) \in S$ with $x \in R$. There exist $c \in C(R)$ and $n \in \text{nil}(R)$ such that $x = c + n$. Since f is epic, $f(c) \in C(S)$ and $f(n) \in \text{nil}(R)$. Hence $s = f(c) + f(n)$ is CN decomposition of s. □

Proposition 2.16. *Let $R = \prod_{i\in I} R_i$ be a direct product of rings. R is CN if and only if R_i is CN for each $i \in I$.*

Proof. We may assume that $I = \{1, 2\}$ and $R = R_1 \times R_2$. Note that $C(R) = C(R_1) \times C(R_2)$ and $nil(R) = nil(R_1) \times nil(R_2)$.
Necessity: Let $r_1 \in R_1$. Then $(r_1, 0) = (c_1, c_2) + (n_1, n_2)$ where $(c_1, c_2) \in C(R)$ and $(n_1, n_2) \in nil(R)$. Hence $r_1 = c_1 + n_1$ is the CN decomposition of $r_1 \in R_1$. So R_1 is CN. A similar proof takes care for R_2 be CN.
Sufficiency: Assume that R_1 and R_2 are CN. Let $(r_1, r_2) \in R$. By assumption r_1 and r_2 have CN decompositions $r_1 = c_1+n_1$ and $r_2 = c_2+n_2$ where c_1 is central in R_1, n_1 is nilpotent in R_1 and c_2 is central in R_2, n_2 is nilpotent in R_2. Hence (r_1, r_2) has a CN decomposition $(r_1, r_2) = (c_1, c_2) + (n_1, n_2)$. This completes the proof. □

Let R be a ring and $D(\mathbb{Z}, R)$ denote the *Dorroh extension* of R by the ring of integers $\mathbb{Z}$ (see [3]). Then $D(\mathbb{Z}, R)$ is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with componentwise addition and multiplication $(n, r)(m, s) = (nm, ns + mr + rs)$ where (n, r), $(m, s) \in D(\mathbb{Z}, R)$. It is clear that $C(D(\mathbb{Z}, R)) = \mathbb{Z} \oplus C(R)$. The identity of $D(\mathbb{Z}, R)$ is $(1, 0)$ and the set of all nilpotent elements is $\text{nil}(D(\mathbb{Z}, R)) = \{(0, r) \mid r \in \text{nil}(R)\}$.

Theorem 2.17. *Let R be a ring. Then R is a CN ring if and only if $D(\mathbb{Z}, R)$ is CN.*

Proof. Assume that R is CN. Let $(a, r) \in D(\mathbb{Z}, R)$. Since R is a CN ring, $r = c + n$ for some $c \in C(R)$ and $n \in \mathrm{nil}(R)$. Then $(a, r) = (a, c) + (0, n)$ is the CN decomposition of (a, r). Conversely, let $r \in R$. Then $(0, r) = (a, c) + (0, s)$ as a CN decomposition where $(n, c) \in C(D(\mathbb{Z}, R))$ and $(0, n) \in \mathrm{nil}(D(\mathbb{Z}, R))$. Then $c \in C(R)$ and $s \in \mathrm{nil}(R)$. It follows that $r = c + s$ is the CN decomposition of r. Hence R is CN. $\square$

Let R be a ring and S a subring of R and

$$T[R, S] = \{(r_1, r_2, \cdots, r_n, s, s, \cdots) : r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}.$$

Then $T[R, S]$ is a ring under the componentwise addition and multiplication. Note that $\mathrm{nil}(T[R, S]) = T[\mathrm{nil}(R), \mathrm{nil}(S)]$ and $C([T, S]) = T[C(R), C(R) \cap C(S)]$.

Proposition 2.18. *R be a ring and S a subring of R. Then the following are equivalent.*

1. *$T[R, S]$ is CN.*
2. *R and S are CN.*

Proof. (1) $\Rightarrow$ (2) Assume that $T[R, S]$ is a CN ring. Let $a \in R$ and $X = (a, 0, 0, \dots) \in T[R, S]$. There exist a central element $C = (r_1, r_2, \cdots, r_n, s, s, \cdots)$ and a nilpotent element $N = (s_1, s_2, \cdots, s_k, t, t, \cdots)$ in $T[R, S]$ such that $X = C + N$. Then r_1 is in the center of R and s_1 is nilpotent in R and $a = r_1 + s_1$ is the CN decomposition of a. Hence R is CN. Let $s \in S$. By considering $Y = (0, s, s, s, \cdots) \in T[R, S]$, it can be seen that s has a CN decomposition.
(2) $\Rightarrow$ (1) Let R and S be CN rings and $Y = (a_1, a_2, \cdots, a_m, s, s, s, \cdots)$ be an arbitrary element in $T[R, S]$. Then there exist $c_i \in C(R)$, $1 \leq i \leq m$, $c \in C(R) \cap C(S)$ and $n_i \in \mathrm{nil}(R)$, $1 \leq i \leq m$, $t \in \mathrm{nil}(S)$ and such that $a_i = c_i + n_i$ for all $1 \leq i \leq m$ and $s = c + t$. Let $C = (c_1, c_2, \cdots, c_m, c, c, \cdots)$ and $N = (n_1, n_2, \cdots, n_m, t, t, \cdots)$. It is obvious that $C \in C(T[R, S])$ and $N \in \mathrm{nil}(T[R, S])$. Hence $Y = C + N$ is a CN decomposition of Y. $\square$

3 Some CN subrings of matrix rings

In this section, we study some subrings of full matrix rings whether or not they are CN rings. We first determine nilpotent and central elements of so-called subrings of matrix rings.

The rings $L_{(s,t)}(R)$ **:** Let R be a ring, and $s, t \in C(R)$. Let $L_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R \right\}$, where the operations are defined as those in $M_3(R)$. Then $L_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 3.1. *Let R be a ring, and let s, t be in the center of R. Then the following hold.*

(1) The set of all nilpotent elements of $L_{(s,t)}(R)$ is

$$nil(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R) \mid a, d, f \in nil(R), c, e \in R \right\}.$$

(2) The set of all central elements of $L_{(s,t)}(R)$ is

$$C(L_{(s,t)}(R))) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid sa = sd, td = tf, a, d, f \in C(R) \right\}.$$

Proof. (1) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in \text{nil}(L_{(s,t)}(R)$. Assume that $A^n = 0$. Then $a^n = d^n = f^n = 0$. Conversely, Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in (L_{(s,t)}(R)$ with $a^{n_1} = 0$, $d^{n_1} = 0$ and $f^{n_1} = 0$ and $n = \max\{n_1, n_2, n_3\}$. Then $A^{n+1} = 0$.

(2) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in C(L_{(s,t)}(R)))$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ s & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \in L_{(s,t)}(R))$.

By $AB = BA$ implies $sc + sd = sa$ and $td = tf$.................................(*)

Let $C = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & t \\ 0 & 0 & 1 \end{bmatrix} \in L_{(s,t)}(R)$).

$AC = CA$ implies $sa = sd$ and $dt + te = tf$.......................................(**).

(*) and (**) implies $sa = sd$ and $tf = td$. For the converse inclusion, let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$ with $sa = sd$, $td = tf$ and a, d, $f \in C(R)$.

Let $B = \begin{bmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{bmatrix} \in L_{(s,t)}(R)$. Then $AB = \begin{bmatrix} ax & 0 & 0 \\ sdy & dz & e \\ 0 & 0 & tdu \end{bmatrix}$, $BA = \begin{bmatrix} xa & 0 & 0 \\ sya & zd & tuf \\ 0 & 0 & vf \end{bmatrix}$. By the conditions; $sa = sd$, $td = tf$, $sc = 0$, $te = 0$ and a, d, $f \in C(R)$, $AB = BA$ for all $B \in L_{(s,t)}(R)$. Hence $A \in C(L_{(s,t)}(R))$. □

Consider following subrings of $L_{(s,t)}(R)$.

$$V_2(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in L_{(s,t)}(R) \mid a, e \in R \right\}$$

$$C(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R) \mid a, d, f \in C(R), c, e \in R, sa = sd, td = tf \right\}$$

It is easy to check that $V_2(L_{(s,t)}(R))$ and $C(L_{(s,t)}(R))$ are subrings of $L_{(s,t)}(R)$.

Proposition 3.2. *Let R be a ring. Following hold:*

(1) R is a CN ring if and only if $V_2(L_{(s,t)}(R))$ is a CN ring.

(2) $C(L_{(s,t)}(R))$ is a ring consisting of elements having CN decompositions.

(3) *Assume that R is a CN ring. If for any $\{a, d, f\} \subseteq R$ having a CN decomposition $a = x+p$, $d = y+q$ and $f = z+r$ with $\{x, y, z\} \subseteq C(R)$ and $\{p, q, r\} \subseteq nil(R)$ satisfy $sx = sy$ and $ty = tz$, then $L_{(s,t)}(R)$ is a CN ring.*

Proof. (1) Assume that R is a CN ring. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in V_2(L_{(s,t)}(R))$. There exist $c \in C(R)$ and $n \in \text{nil}(R)$ such that $a = c + n$. Then $C = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} n & 0 & 0 \\ 0 & n & te \\ 0 & 0 & n \end{bmatrix} \in \text{nil}(V_2(L_{(s,t)}(R)))$ and $A = C + N$ is the CN decomposition of A in $V_2(L_{(s,t)}(R))$. For the inverse implication, let $r \in R$ and consider $A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \in V_2(L_{(s,t)}(R))$. There exist $C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in C(V_2(L_{(s,t)}(R)))$ and $N = \begin{bmatrix} p & 0 & 0 \\ 0 & r & tu \\ 0 & 0 & v \end{bmatrix} \in \text{nil}(V_2(L_{(s,t)}(R)))$. Then $a \in C(R)$ and $p \in \text{nil}(R)$ and $r = a + n$ is the CN decomposition of r. Hence R is a CN ring.

(2) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in CN_{(s,t)}(R)$. Set $C = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 0 \\ sc & 0 & te \\ 0 & 0 & 0 \end{bmatrix}$. By Lemma 3.1, $C \in C(L_{(s,t)}(R))$ and $N \in \text{nil}(L_{(s,t)}(R)$. $A = C + N$ is the CN decomposition of A.

(3) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$. Let $a = x+p$, $d = y+q$ and $f = z+r$ denote the CN decompositions of a, d and f. By hypothesis $sx = sy$ and $ty = tz$. By (2) A has a CN decomposition in $L_{(s,t)}(R)$ as $A = C+N$ where $C = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} p & 0 & 0 \\ sc & q & te \\ 0 & 0 & r \end{bmatrix} \in \text{nil}(L_{(s,t)}(R))$. $\square$

Corollary 3.3. *Let R be a ring. If $L_{(s,t)}(R)$ is a CN ring, then R is a CN ring.*

Proof. Assume that $L_{(s,t)}(R)$ is a CN ring and let $a \in R$ and $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in L_{(s,t)}(R)$. By hypothesis there exist $C = \begin{bmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} n & 0 & 0 \\ sc & m & te \\ 0 & 0 & k \end{bmatrix} \in \text{nil}(L_{(s,t)}(R))$ such that $A = C + N$ where $x \in C(R)$ and $n \in \text{nil}(R)$. Then $a = x + n$ is the CN decomposition of a. □

There are CN rings such that $L_{(s,t)}(R)$ need not be a CN ring.

Example 3.4. Let $R = \mathbb{Z}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \in L_{(1,1)}(R)$. Assume that $A = C + N$ is a CN decomposition of A. Since A is neither central nor nilpotent, by Lemma 3.1, we should get A had a CN decomposition as $A = C + N$ where $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C(L_{(1,1)}(R))$ and $N = \begin{bmatrix} x & 0 & 0 \\ c & y & e \\ 0 & 0 & z \end{bmatrix} \in \text{nil}(L_{(1,1)}(R))$ where $\{x, y, z\} \subseteq \text{nil}(\mathbb{Z})$. This leads us a contradiction in $\mathbb{Z}$.

Proposition 3.5. *R is CN ring if and only if so is $L_{(0,0)}(R)$.*

Proof. Note that $L_{(0,0)}(R)$ is isomorphic to the ring $R \times R \times R$. By Proposition 2.16, $\prod_{i \in I} R_i$ is a CN ring if and only if each R_i is a CN ring for each $i \in I$. □

The rings $H_{(s,t)}(R)$: Let R be a ring and s, t be in the center of R. Let

$$H_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$. Note that any element A of $H_{(s,t)}(R)$ has the form $\begin{bmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{bmatrix}$.

Lemma 3.6. *Let R be a ring, and let s, t be in the center of R. Then the set of all nilpotent elements of $H_{(s,t)}(R)$ is*

$$nil(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid a, d, f \in nil(R), c, e \in R \right\}.$$

Proof. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in \text{nil}(H_{(s,t)}(R))$. There exists a positive integer n such that $A^n = 0$. Then $a^n = d^n = f^n = 0$. Conversely assume that $a^n = 0$, $d^m = 0$ and $f^k = 0$ for some positive integers n, m, k. Let $p = max\{n, m, k\}$. Then $A^{2p} = 0$. □

Lemma 3.7. *Let R be a ring, and let s and t be central invertible in R. Then*

$$C(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid c, e, f \in C(R) \right\}.$$

Proof. [4, Lemma 3.1]. □

Theorem 3.8. *Let R be a ring. R is a CN ring if and only if $H_{(s,t)}(R)$ is a CN ring.*

Proof. Assume that R is a CN ring. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in (H_{(s,t)}(R))$. Then $a = c_1 + n_1$, $d = c_2 + n_2$, $f = c_3 + n_3$, $c = c_4 + n_4$, $e = c_5 + n_5$ with $\{c_1, c_2, c_3, c_4, c_5\} \subseteq C(R)$, $\{n_1, n_2, n_3, n_4, n_5\} \subseteq \text{nil}(R)$. Let $c_1 - c_2 = sc_4$, $c_2 - c_3 = tc_5$, $n_1 - n_2 = sn_4$ and $n_2 - n_3 = tn_5$ and $C = \begin{bmatrix} c_1 & 0 & 0 \\ c_4 & c_2 & c_5 \\ 0 & 0 & c_3 \end{bmatrix}$ and

$N = \begin{bmatrix} n_1 & 0 & 0 \\ n_4 & n_2 & n_5 \\ 0 & 0 & n_3 \end{bmatrix}$. By Lemma 3.7, $C \in C(H_{(s,t)}(R))$ and by Lemma 3.6, $N \in \mathrm{nil}(H_{(s,t)}(R))$. Then $A = C + N$ is the CN decomposition of A.

Conversely, suppose that $H_{(s,t)}(R)$ is a CN ring. Let $a \in R$. Then $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in (H_{(s,t)}(R))$ and it has a CN decomposition $A = C + N$ where $C = \begin{bmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{bmatrix} \in C(H_{(s,t)}(R))$ with $\{y, u, v\} \subseteq C(R)$ and $N = \begin{bmatrix} n_1 & 0 & 0 \\ n_2 & n_3 & n_4 \\ 0 & 0 & n_5 \end{bmatrix} \in \mathrm{nil}(H_{(s,t)}(R))$ with $\{n_1, n_3, n_5\} \subseteq \mathrm{nil}(R)$. Then $a = x + n_1$ is a CN decomposition of a. □

Proposition 3.9. *Uniquely nil clean rings, uniquely strongly nil clean rings, strongly nil *-clean rings are CN.*

Proof. These classes of rings are abelian. Assume that R is uniquely nil clean ring. Let e be an idempotent in R. For any $r \in R$, $e + (re - ere)$ can be written in two ways as a sum of an idempotent and a nilpotent as $e+(re-ere) = (e+(re-ere))+0 = e+(re-ere)$. Then $e = e+(re-ere)$ and $er - ere = 0$. Similarly, $e+(er-ere) = (e+(re-ere))+0 = e+(er-ere)$. Then $0 = re - ere = re - ere$. Hence e is central. □

The converse of this result is not true.

Example 3.10. *The ring $H_{(0,0)}(\mathbb{Z})$ is CN but not uniquely nil clean.*

Proof. By Theorem 3.8, $H_{(0,0)}(\mathbb{Z})$ is CN. Note that for $n \in \mathbb{Z}$ has a uniquely nil clean decomposition if and only if $n = 0$ or $n = 1$. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & a & e \\ 0 & 0 & a \end{bmatrix} \in H_{(0,0)}(R)$ with $a \notin \{0, 1\}$. Assume that A has a uniquely nil clean decomposition. There exist unique $E^2 = E = \begin{bmatrix} x & 0 & 0 \\ y & x & u \\ 0 & 0 & x \end{bmatrix} \in H_{(0,0)}(R)$ and

$N = \begin{bmatrix} g & 0 & 0 \\ h & g & l \\ 0 & 0 & g \end{bmatrix} \in N(H_{(0,0)}(R)$ such that $A = E + N$. Then A has a uniquely nil clean decomposition. So $a = x + g$ has a CN decomposition. This is not the case for $a \in \mathbb{Z}$. Hence $H_{(0,0)}(\mathbb{Z})$ is not uniquely nil clean. $\square$

Generalized matrix rings: Let R be ring and s a central element of R. Then $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and with multiplication defined in [6] by

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{bmatrix}.$$

In [6], $K_s(R)$ is called a *generalized matrix ring over R.*

Lemma 3.11. *Let R be a commutative ring. Then the following hold.*

(1) $nil(K_0(R)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(R) \mid \{a, d\} \subseteq nil(R) \right\}$.

(2) $C(K_0(R))$ *consists of all scalar matrices.*

Proof. (1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{nil}(K_0(R))$. Then $A^2 = \begin{bmatrix} a^2 & b(a+d) \\ c(a+d) & d^2 \end{bmatrix}$, ...,

... , $A^{2^n} = \begin{bmatrix} a^{2^n} & \Sigma_{i=1}^{n} b(a^{2^{i-1}} + d^{2^{i-1}}) \\ \Sigma_{i=1}^{n} c(a^{2^{i-1}} + d^{2^{i-1}}) & d^{2^n} \end{bmatrix}$. Hence $A \in \mathrm{nil}(K_0(R))$ if and only if $\{a, d\} \subseteq \mathrm{nil}(R)$. $\square$

Lemma 3.12. *Let R be ring. Then R is a CN ring if and only if* $D_n(K_0(R))$ *is a CN ring.*

Proof. Necessity: We assume that $n = 2$. Let $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in D_2((K_0(R)))$. By assumption $a = c_1 + n_1$ where $c_1 \in C(R)$ and $n_1 \in \mathrm{nil}(R)$. Let $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} \in C(D_2(K_0(R)))$ and $N = \begin{bmatrix} n_1 & b \\ 0 & n_1 \end{bmatrix} \in \mathrm{nil}D_2((K_0(R)))$. $A = C+N$ is the CN decomposition of A.

Sufficiency: Let $a \in R$. Then $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in D_2((K_0(R)))$ has a CN decomposition $A = C + N$ with $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} \in C(D_2((K_0(R))))$ and $N = \begin{bmatrix} n_1 & b_1 \\ 0 & n_1 \end{bmatrix} \in \text{nil}(D_2((K_0(R))))$ where $c_1 \in C(R)$ and $n_1 \in \text{nil}(R)$. By comparing components of matrices we get $a = c_1 + n_1$. It is a CN decomposition of a. $\square$

Note that $K_0(R)$ need not be a CN ring.

Example 3.13. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(\mathbb{Z})$ have a CN decomposition as $A = C + N$ where $C \in C(K_0(\mathbb{Z}))$ and $N \in \text{nil}K_0(\mathbb{Z}))$. Then we should have $C = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ and $N = \begin{bmatrix} 1-x & 0 \\ 0 & -x \end{bmatrix}$. These imply $x = 1$ or x is nilpotent. A contradiction.

References

[1] H. Chen, S. Halicioglu, A. Harmanci and Y. Kurtulmaz,*On the Decomposition of a Matrix into the Sum of a Central Matrix and a Unit Matrix* , submitted for publication.

[2] A. J. Diesl, *Nil clean rings*, J. Algebra, 383(2013), 197-211.

[3] J. L. Dorroh, *Concerning Adjunctions to Algebras*, Bull. Amer. Math. Soc., 38(1932), 85-88.

[4] H. Kose, Y. Kurtulmaz, B. Ungor and A. Harmanci, *Rings Having Normality in terms of the Jacobson Radical*, Arab. J. Math., https://doi.org/10.1007/s40065-018-0231-7.

[5] Y. Hirano, H. Tominaga and A. Yaqub, *On rings in which every element is uniquely expressable as a sum of a nilpotent element and a certain potent element*, Math. J. Okayama Univ. 30(1988), 33-40.

[6] P. A. Krylov, *Isomorpism of generalized matrix rings*, Algebra and Logic, 47(4)(2008), 258-262.

[7] C. Li and Y. Zhou, *On strongly *-clean rings*, J. Algebra Appl. 10(2011), 1363-1370.